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Effect of the Earth's Magnetic Field on the Motion of an Artificial Satellite

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The effect of the earth's magnetic field on the motion of an artificial satellite moving in a circular orbit in the plane of the magnetic equator of the earth is discussed. Approximate formulas for the current induced in the satellite and for the resulting induction drag are obtained. The current in a conducting satellite of an average size at an altitude of several hundred kilometers is estimated to be of the order of milliamperes. The induction drag may exceed the friction drag for satellites of large dimensions and for elongated satellites.

I. INTRODUCTION

WHENEVER a conductor moves through a conducting medium in the presence of a magnetic field that has a component normal to the direction of the motion, an electric current is induced in the conductor, and the conductor experiences an induction drag.[1] Since the upper ionosphere and the interplanetary space are conducting media due to the presence of free electrons and ions in them, it is apparent that an electric current may be induced in a conducting body moving in the upper ionosphere or in the interplanetary space, and that this body may experience an induction drag whenever it traverses a magnetic field in its path.

It is the purpose of this paper to determine the current induced in an artificial satellite by the earth's magnetic field, to evaluate the resulting drag, and to investigate the effect of this drag on the satellite's motion for the simple case of a spherical satellite moving without spin in a circular orbit in the plane of the magnetic equator of the earth. The calculations of the induced current are carried out under the assumption of the following idealized conditions: (1) the velocity of the satellite is negligibly small compared with the velocity of light; (2) the velocity of the satellite is essentially constant; (3) all points of the satellite have essentially the same velocity; (4) the magnetic field in the satellite's orbit is constant; (5) the induced current does not affect the magnetic field; (6) the material of the satellite and the medium in the satellite's orbit are linear and isotropic conductors; (7) the conductivity of the medium in the neighborhood of the satellite is constant; (8) the medium in the neighborhood of the satellite is stationary.

II. THEORY

The fundamental steady-state field equations which are valid in the stationary as well as in the moving nonmagnetized, nonpolarized media are the two Maxwell's equations

$$\nabla\times\mathbf{E}=0 \tag{1}$$

and

$$\nabla\times\mathbf{H}=\mathbf{J}+\rho\mathbf{v}, \tag{2}$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}$, and $\rho\mathbf{v}$ are the electric field vector, the magnetic field vector, the conduction current density vector, and the convection current density vector, respectively, all measured in a stationary frame of reference.[2] If the velocity of motion is small compared with the velocity of light, and only an induced electric field is present, the magnetic field vector and the conduction current density vector may be considered the same in both the moving and the stationary frame of reference, while the electric field vector in the moving frame of reference may be expressed in terms of the quantities measured in the stationary frame of reference as

$$\mathbf{E}^*=\mathbf{E}+\mathbf{v}\times\mathbf{B}, \tag{3}$$

where the asterisk indicates the vector measured in the moving frame of reference.

If both the moving and the stationary media are linear and isotropic conductors, the conduction current density is given by Ohm's law,

$$\mathbf{J}=\sigma_s\mathbf{E} \tag{4}$$

[1] See T. G. Cowling, "Solar electrodynamics," in G. K. Kuiper, editor, *The Sun* (University of Chicago Press, Chicago, 1953).

[2] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 147.

for the stationary medium and

$$\mathbf{J}=\sigma_m\mathbf{E}^*=\sigma_m(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{5}$$

for the moving medium, where σ_s and σ_m are the conductivities of the stationary and the moving medium, respectively.

With the current density in the moving medium determined from these equations, subject to the geometry of the problem, the magnetic drag experienced by this medium can be evaluated with the aid of the equation

$$\mathbf{F}=\int\mathbf{J}\times\mathbf{B}d\tau, \tag{6}$$

where the integral is extended over the volume of the moving medium.

For a satellite (moving medium) moving with a constant velocity $\mathbf{v}$ through a constant magnetic field $\mathbf{B}$ the problem of calculating the current density may be reduced to an equivalent electrostatic problem. Indeed, for the frame of reference moving with the satellite, Eqs. (1) and (3) give in this case,

$$\nabla\times\mathbf{E}^*=0, \tag{7}$$

and, therefore,

$$\nabla\times(\mathbf{E}+\mathbf{v}\times\mathbf{B})=\nabla\times\mathbf{E}=0,$$

so that $\mathbf{E}$ may be expressed as

$$\mathbf{E}=-\nabla\varphi. \tag{8}$$

Furthermore, since the convection current disappears in this frame of reference, and since $\mathbf{v}$ and $\mathbf{B}$ are constant, Eqs. (2), (4) or (5), and (7) give for both the stationary and the moving medium

$$\nabla\cdot\nabla\times\mathbf{H}^*=\nabla\cdot\mathbf{J}^*=\nabla\cdot\mathbf{J}=-\sigma\nabla^2\varphi=0.$$

Thus the potential φ everywhere satisfies Laplace's equation,

$$\nabla^2\varphi=0, \tag{9}$$

with the usual boundary conditions for the potential,

$$\varphi_s=\varphi_m, \tag{10}$$

and for the normal component of the current density,

$$J_{ns}=J_{nm}, \tag{11}$$

on the boundary between the moving (m) and the stationary medium (s). At infinity the potential φ must be zero.[3]

[3] This last condition would not be true if φ were defined as the potential of the field $\mathbf{E}^*$ rather than the field $\mathbf{E}$.

III. CALCULATION OF CURRENT AND DRAG

Let a right-handed system of rectangular coordinates be placed in the center of the satellite so that the x axis lies in the direction of motion of the satellite, while the y axis lies in the direction of the earth's magnetic field. Let the z axis also serve as the polar axis of spherical coordinates with the same origin, so that the polar angle θ is the angle subtended by the z axis and the radius vector $\mathbf{r}$ at the center of the satellite. For a hollow spherical satellite with internal radius a and external radius b, the boundary conditions (10) and (11) assume then the form

$$\varphi_1=\varphi_2,\quad 0=-\frac{\partial\varphi_2}{\partial r}+vB\cos\theta, \tag{12a, b}$$

at $r=a$, and

$$\varphi_2=\varphi_3,\quad -\sigma_2\frac{\partial\varphi_2}{\partial r}+\sigma_2 vB\cos\theta=-\sigma_3\frac{\partial\varphi_3}{\partial r}, \tag{13a, b}$$

at $r=b$, where the subscripts 1, 2, and 3 refer to the potentials and conductivities in the regions with $r<a$, $a<r<b$, and $r>b$, respectively (σ_1 is assumed to be zero).

As one can see, the solutions to Laplace's equation (9) remaining finite at the center of the satellite as well as at infinity may be written for the three regions under consideration as

$$\varphi_1=C_1 r\cos\theta, \tag{14}$$

$$\varphi_2=C_2\left(1+C_3\frac{a^3}{r^3}\right)r\cos\theta, \tag{15}$$

$$\varphi_3=C_4\frac{\cos\theta}{r^2}, \tag{16}$$

where C_1, C_2, C_3, and C_4 are constants which may be evaluated with the aid of the boundary conditions (12) and (13). Their values are readily found to be

$$C_1=\frac{vB(\sigma_2-\sigma_3)(b^3-a^3)}{(\sigma_2+2\sigma_3)(b^3-a^3)+3\sigma_3a^3},$$

$$C_2=\frac{vB[\sigma_2(b^3-a^3)+\sigma_3a^3]}{(\sigma_2+2\sigma_3)(b^3-a^3)+3\sigma_3a^3},$$

$$C_3=\frac{\sigma_3b^3}{\sigma_2(b^3-a^3)+\sigma_3a^3},$$

$$C_4=\frac{\sigma_2vB(b^3-a^3)b^3}{(\sigma_2+2\sigma_3)(b^3-a^3)+3\sigma_3a^3}.$$

The J_z component of the induced current density in the satellite is, according to Eqs. (5), (8), and (15),

$$J_z = -\sigma_2 \frac{\partial \varphi_2}{\partial z} + \sigma_2 vB$$
$$= \sigma_2 \left\{ vB - C_2 \left[1 + C_3 \frac{a^3}{r^3} (1 - 3\cos^2\theta) \right] \right\}. \quad (17)$$

The total current obtained after integrating this expression over the cross section of the satellite in x, y plane is

$$I = \frac{2\sigma_2\sigma_3 b^2 (b-a)(b^2+ab-a^2)}{(\sigma_2+2\sigma_3)(b^3-a^3)+3\sigma_3 a^3} \pi vB. \quad (18)$$

This formula can be considerably simplified for the special cases of a solid satellite, $a=0$, and of a thin-walled satellite, $a/b = 1-\delta$, where δ is a small number. It can be simplified still more if the conductivity of the satellite is much larger than the conductivity of the surrounding space, $\sigma_2 \gg \sigma_3$, which very likely is the case for metallic satellites. For a solid satellite Eq. (18) becomes in this case

$$I = 2\sigma_3 b^2 \pi vB. \quad (19)$$

For a thin-walled satellite it becomes

$$I = \tfrac{2}{3}\sigma_3 b^2 \pi vB. \quad (20)$$

(The last formula is true only if $\sigma_2\delta \gg \sigma_3$, which excludes satellites with extremely thin walls.)

The induction drag exerted upon the satellite by the earth's magnetic field may be evaluated with the aid of Eq. (6), which for the present choice of coordinates becomes

$$\mathbf{F} = \int \mathbf{J} \times \mathbf{B} d\tau = -\mathbf{i} \int J_z B_y d\tau = -\mathbf{i} \int J_z B d\tau, \quad (21)$$

where the integral is to be taken over the volume of the spherical shell with internal radius a and external radius b. The substitution of Eq. (17) into the last integral of Eq. (21) yields after integration and simplification

$$\mathbf{F} = -\mathbf{i} \frac{2\sigma_2\sigma_3 b^3 vB^2 \tau}{(\sigma_2+2\sigma_3)(b^3-a^3)+3\sigma_3 a^3}, \quad (22)$$

where τ is the volume of the satellite's shell.

The acceleration resulting from the magnetic drag is then

$$\mathbf{f} = -\mathbf{i} 2\sigma_3 \frac{vB^2}{d} \quad (23)$$

and

$$\mathbf{f} = -\mathbf{i} 2\sigma_3 \frac{vB^2}{3\delta d}, \quad (24)$$

for a solid and an empty thin-walled satellite, respectively, if $\sigma_2\delta \gg \sigma_3$ (d is the density of the satellite's material).

IV. CONDUCTIVITY OF SPACE

The derivations of the preceding sections were based on the assumption that both the satellite and the surrounding space are macroscopic systems which may be characterized by their respective conductivities. Conductivity, however, is normally defined only for systems in which the mean free path of the charge carriers is much smaller than the dimensions of the field inhomogeneities. As far as the material of the satellite is concerned, this mean free path requirement is of course well satisfied. In the space surrounding the satellite, however, the mean free path of the charge carriers is of the order of kilometers, which greatly exceeds the dimensions of both the satellite and the inhomogeneities of the satellite's field (the field due to charges induced on the surface of the satellite). Thus, as far as the surrounding space is concerned, the system under consideration may be regarded as a microscopic one, the satellite being merely an abnormally large polar molecule surrounded by the particles of the ionospheric gas. Consequently, the problem solved in the preceding sections must be regarded as a "macroscopic approximation" of the actual problem. The conductivity of space σ_3 must, therefore, be understood as an "effective conductivity," that is, as the conductivity of a conductor which would cause approximately the same current in a satellite moving through it, as the current caused in the satellite by ionospheric particles through which the satellite is actually moving. In order to determine the magnitude of this conductivity it is obviously necessary to consider the interactions between the satellite and the surrounding particles, or, in other words, to consider the problem of the orbiting satellite in its "microscopic approximation."

From the microscopic point of view the drag experienced by the satellite is the result of the transfer of momentum from the surrounding particles to the satellite. For the approaching particles the satellite constitutes a scattering center. Its scattering cross section may be determined by noticing that only a particle whose kinetic energy is smaller than its potential energy in the field of the satellite may transfer an appreciable part of its momentum to the satellite (not counting the direct hits).[1] From large distances the satellite may be regarded as an electric dipole with a dipole moment approximately equal to $4\pi\epsilon_0 b^3 vB$. With this expression for the dipole moment the scattering cross section is readily found to be

$$S=2\pi\left(\frac{q}{m}\right)\frac{Bb^3}{v}=\frac{3}{2}\left(\frac{q}{m}\right)\frac{B\tau}{v}, \tag{25}$$

where q/m is the charge-mass ratio of the particles, and the remaining symbols are the same as before (the thermal motion and the effect of the earth's magnetic field on the trajectories of the particles are neglected).

The magnitude of the drag is equal to the rate of momentum transfer from the oncoming particles to the satellite. Taking into account the differences in the scattering cross sections for different particles, the drag may be expressed as

$$F=\frac{\Delta G}{\Delta t}=v^2\sum(S_iN_im_i), \tag{26}$$

where ΔG is the momentum transfered, Δt is the time, S_i is the scattering cross section for the particles of type i, N_i is the number density of these particles, and m_i is the mass of each of these particles. Combining Eqs. (25) and (26) we now find that the acceleration experienced by the satellite is approximately given by

$$f=\tfrac{3}{2}\sum(q_iN_i)\frac{Bv}{d}. \tag{27}$$

Comparing Eq. (27) with Eq. (23) we finally obtain for the effective value of the conductivity of space

$$\sigma_3\approx\sum(q_iN_i)B^{-1}. \tag{28}$$

V. DISCUSSION

It may be assumed that in the upper ionosphere (about 300 km above the earth's surface) the density of the charge carriers is of the order of 10^{12} m^{-3}, the number of electrons is approximately equal to the number of ions (each having a charge of 1.6×10^{-19} amp·sec), and the flux density of the earth's magnetic field is of the order of 10^{-5} v·sec/m^2.[4] The effective conductivity of space is in this case, according to Eq. (28),

$$\sigma_3\approx2\times10^{-2}\frac{\text{amp}}{\text{v}\cdot\text{m}}.$$

With this value for the conductivity the total current induced in a spherical satellite of 1 m^2 cross-sectional area moving with a velocity of about 8×10^3 m/sec is, according to Eq. (19),

$$I\approx3\times10^{-3}\text{ amp}.$$

Thus the current induced in a satellite by the earth's magnetic field may have an appreciable magnitude and may possibly affect the functioning of high-sensitivity instruments carried by the satellite. The measurement of the induced current may be desirable for an accurate evaluation of the induction drag.

The significance of the induction drag as a factor perturbing the motion of a satellite depends on the relative magnitude of this drag in comparison with the friction drag. The friction drag may be estimated with the aid of Eq. (26) if all the collision cross sections S_i are set equal to the cross-sectional area of the satellite, and the summation is taken over *all* particles impinging on the satellite. The ratio of the induction to the friction drag may, therefore, be written, according to Eqs. (26) and (27), as

$$\frac{F_{\text{in}}}{F_{\text{fr}}}=\frac{2\sum(q_iN_i)}{\sum(N_nm_n)}\cdot\frac{Bb}{v}, \tag{29}$$

where the sum in the numerator is to be taken over the charged particles only, while the sum in the denominator is to be taken over all particles. Thus, the relative significance of the induction drag increases with increasing percentage of ionization, and, therefore, this drag becomes

[4] See, for instance, Karl Rawer, *The Ionosphere* (Fredrick Ungar Publishing Company, New York, 1957).

especially important in the uppermost regions of the ionosphere. Furthermore, the relative significance of the induction drag increases with the size of the satellite. In this connection it is interesting to note that, since the induction drag is essentially proportional to the *volume* of the satellite, while the friction drag is essentially proportional to the *cross-sectional area* of the satellite, the ratio of the induction to the friction drag is essentially proportional to the length of the satellite.

The induction drag may be considered significant if it comprises 1% or more of the total drag. In the uppermost regions of the ionosphere (above 500 km), the number of neutral praticles may be assumed to be of the same order as the number of charged particles, and the average atomic weight of these particles may be assumed to be about 16 (essentially oxygen). In this case Eq. (29) shows that the radius (length) of a satellite whose induction drag comprises more than 1% of the total drag is of the order of 0.5 m. Neglecting the induction drag may, therefore, result in too high a value for atmospheric density at very high altitudes if the density is estimated from the perturbations of the satellite's motion. The errors may be especially large if the satellite is long and if the average atomic weight of the particles in the satellite's orbit is small.

In conclusion, it may be added that the induction effect discussed in this paper is only one of several electrodynamical effects which may influence a planetary motion.[5] These effects may be of considerable cosmological significance and may be responsible for certain peculiarities observed in the structure and mechanics of the solar system. An experimental study of these effects with the aid of artificial satellites is therefore highly desirable.

[5] Related effects have been recently discussed in K. P. Chopra, J. Geophys. Research **62**, 143 (1957) and R. Jastrow and C. A. Pearse, J. Geophys. Research **62**, 413 (1957).